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Sensitivity analysis for abstract equilibrium problems

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Abstract

We develop the general framework of sensitivity analysis for equilibrium problems in the setting of vector topological normed space. Our approach does not make any recourse to geometrical properties and the obtained result can be viewed as an extension and generalization of the well-known results (on variational inequalities) in the literature. Even though we have worked under arbitrary constraints $\mathcal{K}_\lambda$ with Hölder-property—that have been decisive in our treatment—we have obtained, in a similar spirit of Domokos [J. Math. Anal. Appl. 230 (1999) 382–389], the best lower bound for the continuity modulus despite of the properties of the boundary of $\mathcal{K}_\lambda$.

Keywords: Parametric equilibrium problems; Sensitivity analysis

1. Introduction

In recent years, much attention has been given to study equilibrium problems arising in economics, engineering, and financial optimization. It has been shown in [1,2] that equilibrium problems include variational inequalities, quasi-variational inequalities, and

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complementarity problems as special cases. The behavior of such equilibrium solutions as a result of changes in the problem data is always of concern. The motivation to analyze this behavior of solutions when some parameters perturb the problem has came from practical interests and has led to so-called solution sensitivity. In [3], Dafermos investigated sensitivity analysis of a special parametric variational inequalities. Mukherjee and Verma [9] analyzed the class of Noor from sensitivity point of view by means of Dafermos techniques. Rather recently, Moudafi and Noor [8] and Noor [12] have studied sensitivity for variational inclusions and mixed variational inequalities by use of resolvent equations. This technique has been extended by Noor and Noor [14] to study the sensitivity of quasi-variational inclusions.

However, still there are some other interesting and challenging problems which do not fit in the variational inequalities formulation like hemivariational inequalities. The present paper thus tries to build a unifying and general statement of the above contributions by considering the equilibrium problem: given an arbitrary set $\mathcal{K}$ and a real-valued bifunction φ on $\mathcal{K}$, we seek $\bar{u} \in \mathcal{K}$ such that

$$\varphi(\bar{u}, v) \geqslant 0, \quad \forall v \in \mathcal{K}. \tag{EP}$$

For the applications and existence results, we refer among others to the papers [1,2] and references therein. Our aim here is to tackle the sensitivity analysis question for this important formulation in a general setting.

Let us now describe in details the contents of the paper. In Section 2, we first formulate the parametric form of our problem, present our fundamental result, Theorem 2.2.1, and analyze the assumptions subject to our treatment.

The result of this paper unifies and improves many previous results (on the continuity of solutions to variational inequalities) in the literature [3–9,11,12,14–18,20–22], where no appeal is made to geometrical properties of the space. Even though we have worked under arbitrary constraints $\mathcal{K}_\lambda$ with Hölder-property that have been decisive in our treatment—we have obtained, in similar spirit of Domokos [4], the best lower bound for the continuity modulus despite of the properties of the boundary of $\mathcal{K}_\lambda$.

2. The main result

2.1. Parametric form of equilibrium problem

Let X be a normed vector topological space with norm $\|.\|$. Let also M and Λ be subsets of two normed spaces with norms also denoted by $\|.\|$ and $\{\mathcal{K}_\lambda\}_{\lambda \in M}$ be a family of arbitrary subsets of X. We further consider a family of bifunctions $\{\varphi(\,.\,,.\,,\mu)\}_{\mu \in \Lambda}$ defined on $X \times X$. Given a pair $(\bar{\mu}, \bar{\lambda}) \in \Lambda \times M$, we consider the equilibrium problem: find $u(\bar{\mu}, \bar{\lambda}) := \bar{u} \in \mathcal{K}_\lambda$ such that

$$\varphi(\bar{u}, v, \bar{\mu}) \geqslant 0, \quad \forall v \in \mathcal{K}_{\bar{\lambda}}. \tag{$EP_{\bar{\mu},\bar{\lambda}}$}$$

The perturbed form of $(EP_{\bar{\mu},\bar{\lambda}})$ will hence be as follows: find $u(\mu, \lambda) \in \mathcal{K}_\lambda$ such that

$$\varphi\big(u(\mu, \lambda), v, \mu\big) \geqslant 0, \quad \forall v \in \mathcal{K}_\lambda. \tag{$EP_{\mu,\lambda}$}$$

Remark 2.1.1. Note that in [1,2] are provided conditions ensuring the existence of solutions to $(EP_{\mu,\lambda})$. We hence assume from now on, for $\bar{\mu} \in \Lambda$ and $\bar{\lambda} \in M$, that the problem $(EP_{\bar{\mu},\bar{\lambda}})$ admits at least a solution $\bar{u}$ and for each $\mu \in \Lambda \cap \bar{U}$, $\lambda \in M \cap \bar{V}$, the problem $(EP_{\mu,\lambda})$ admits at least a solution $u(\mu,\lambda)$. Here $\bar{U}$ is a neighborhood of $\bar{\mu}$ and $\bar{V}$ is a neighborhood of $\bar{\lambda}$.

To carry out our sensitivity analysis around $\bar{u}$, for each μ and μ' in $\bar{U}$ of $\bar{\mu}$ such and λ, λ' in $\bar{V}$, we make the following assumptions:

(H_0) *Hölder property*: $\mathcal{K}(\lambda) = \mathcal{K}_\lambda$ is Hölder at $\bar{\lambda}$ that is for a neighborhood $\bar{M}$ of $\bar{\lambda}$ and some constant $L > 0$ such that

$$\mathcal{K}_\lambda \subset \mathcal{K}_{\lambda'} + L\|\lambda - \lambda'\|^\xi B_X \quad \text{for all } \lambda, \lambda' \in \bar{M}.$$

Here B_X stands for the unit ball of X.

(H_1) *Strong monotonicity condition*: assume that φ satisfies

$$\varphi(u, v, \mu) + \varphi(v, u, \mu) \leqslant -m\|u - v\|^\alpha$$

for some $m > 0$, $\alpha > 0$, all $u \in \bar{U}$ and all $u, v \in X$.

(H_2) *Lipschitz behavior with respect to second argument*: there exists a neighborhood N of $\bar{\mu}$ and a constants $R > 0$, $\beta > 0$ such that for all $\mu \in N$, $u, v, v' \in X$ we have

$$\left|\varphi(u, v, \mu) - \varphi(u, v', \mu)\right| \leqslant R\|v - v'\|^\beta.$$

(H_3) *Lipschitz property with respect to parameter* μ: there exists $\theta > 0$ and $\gamma > 0$, $\delta > 0$ such that

$$\left|\varphi(u, v, \mu) - \varphi(u, v, \mu')\right| \leqslant \theta\|\mu - \mu'\|^\gamma \|v - u\|^\delta$$

for all $u, v \in X$ and all μ, μ' in a neighborhood of $\bar{\mu}$.

(H_4) *Control of data*: assume that $\alpha > \delta$.

Remark 2.1.2.

- Notice here that assumption (H_1) ensures uniqueness of solutions to (EP) and $(EP_{\mu,\lambda})$.
- if we denote, for each $\lambda \in M$, by I_λ the indicator function of $\mathcal{K}_\lambda$, i.e., the function which takes the value 0 on $\mathcal{K}_\lambda$ and the value $+\infty$ out of $\mathcal{K}_\lambda$. Let us mention that if we replace φ in $(EP_{\mu,\lambda})$ by $\varphi + \Psi_\mu$ with Ψ_λ given by

  $$\Psi_\lambda(u, v) = I_\lambda(v) - I_\lambda(u),$$

  the inequality of the problem $(EP_{\mu,\lambda})$ comes back to

  $$\varphi\big(u(\mu, \lambda), v, \mu\big) + \Psi_\lambda\big(u(\mu, \lambda), v\big) \geqslant 0, \quad \forall v \in X.$$

2.2. The result

Let us now state our main result.

Theorem 2.2.1. *Suppose that assumptions $(H_i)_{i=0,\dots,4}$ hold; then there exist constants $k_1, k_2 > 0$, and neighborhoods U_1 of $\bar{\mu}$, V_1 of $\bar{\lambda}$ such that*

(i) *For each $(\mu, \lambda) \in (\Lambda \cap U_1) \times (M \cap V_1)$ the solution $u(\mu, \lambda)$ to $(EP_{\mu,\lambda})$ is unique.*
(ii) *For all $(\mu, \lambda), (\mu', \lambda') \in (\Lambda \cap U_1) \times (M \cap V_1)$ we have*

$$\left\| u(\mu, \lambda) - u(\mu', \lambda') \right\| \leqslant k_1 \|\mu - \mu'\|^{\gamma/\alpha - \delta} + k_2 \|\lambda - \lambda'\|^{\beta\xi_1/\alpha}.$$

Proof. We organize the proof in three steps.

Step I. Let $\lambda \in \bar{M}$. We shall prove that there exists a neighborhood U_1 of $\bar{\mu}$ and a constant $k_1 > 0$ such that for all $\mu, \mu' \in U_1$ we have

$$\left\| u(\mu, \lambda) - u(\mu', \lambda) \right\| \leqslant k_1 \|\mu - \mu'\|^{\gamma/\alpha - \delta}.$$

Let U_1 be a neighborhood of $\bar{\mu}$ such that (H_1), (H_2), and (H_3) are satisfied for all $\mu, \mu' \in U_1$. Since, for $\kappa \in \{\mu, \mu'\}$, $u(\kappa, \lambda)$ is a solutions to $(EP_{\kappa,\lambda})$, for all $v \in \mathcal{K}_\lambda$, we have

$$\varphi\big(u(\kappa, \lambda), v, \kappa\big) \geqslant 0. \tag{1}$$

Thus, if we make in (1) $\kappa = \mu$ and $v = u(\mu', \lambda)$ (respectively $\kappa = \mu'$ and $v = u(\mu, \lambda)$), by adding the two obtained relations, we obtain

$$\varphi\big(u(\mu, \lambda), u(\mu', \lambda), \mu\big) + \varphi\big(u(\mu', \lambda), u(\mu, \lambda), \mu'\big) \geqslant 0. \tag{2}$$

Therefore,

$$\begin{aligned} 0 \leqslant{}& \varphi\big(u(\mu, \lambda), u(\mu', \lambda), \mu\big) + \varphi\big(u(\mu', \lambda), u(\mu, \lambda), \mu\big) \\ &+ \varphi\big(u(\mu', \lambda), u(\mu, \lambda), \mu'\big) - \varphi\big(u(\mu', \lambda), u(\mu, \lambda), \mu\big). \end{aligned}$$

Using thus (H_1) and (H_3), we conclude that

$$m\left\| u(\mu, \lambda) - u(\mu', \lambda) \right\|^\alpha \leqslant \theta \|\mu - \mu'\|^\gamma \left\| u(\mu, \lambda) - u(\mu', \lambda) \right\|^\delta;$$

therefore, (H_4) drives the last inequality to

$$\left\| u(\mu, \lambda) - u(\mu', \lambda) \right\| \leqslant (\theta/m)^{1/(\alpha-\delta)} \|\mu - \mu'\|^{\gamma/\alpha - \delta}. \tag{3}$$

Step II. Now we shall seek a neighborhood V_1 of $\bar{\lambda}$ and a constant $k_2 > 0$ such that for each $\mu \in U_1$ and all $\lambda, \lambda' \in V_1$ we have

$$\left\| u(\mu, \lambda) - u(\mu, \lambda') \right\| \leqslant k_2 \|\lambda - \lambda'\|^{\beta\xi/\alpha}.$$

To this end take $V_1 = \bar{M}$. Thanks to assumption (H_0), for all $\lambda, \lambda' \in V_1$ there exists $u_1 \in \mathcal{K}_\lambda$ and $u_2 \in \mathcal{K}_{\lambda'}$ such that

$$\left\| u(\mu, \lambda') - u_1 \right\| \leqslant L \|\lambda - \lambda'\|^\xi \quad \text{and} \quad \left\| u(\mu, \lambda) - u_2 \right\| \leqslant L \|\lambda - \lambda'\|^\xi. \tag{4}$$

Keeping this in mind, let us remark that (H_1) leads to

$$m\left\| u(\mu, \lambda) - u(\mu, \lambda') \right\|^\alpha \leqslant -\varphi\big(u(\mu, \lambda), u(\mu, \lambda'), \mu\big) \tag{5}$$

$$- \varphi\big(u(\mu, \lambda'), u(\mu, \lambda), \mu\big). \tag{6}$$

On the other hand, since for $\nu \in \{\lambda, \lambda'\}$, $u(\mu, \nu)$ is a solution to $(EP_{\mu,\nu})$, it follows that

$$\varphi\big(u(\mu, \lambda), u_1, \mu\big) \geqslant 0 \quad \text{and} \quad \varphi\big(u(\mu, \lambda'), u_2, \mu\big) \geqslant 0. \tag{7}$$

Adding the three previous inequalities, we get

$$m\big\|u(\mu, \lambda) - u(\mu, \lambda')\big\|^{\alpha} \leqslant \varphi\big(u(\mu, \lambda'), u_2, \mu\big) - \varphi\big(u(\mu, \lambda'), u(\mu, \lambda), \mu\big) + \varphi\big(u(\mu, \lambda), u_1, \mu\big) - \varphi\big(u(\mu, \lambda), u(\mu, \lambda'), \mu\big).$$

According to (H_2), it follows that

$$m\big\|u(\mu, \lambda) - u(\mu, \lambda')\big\|^{\alpha} \leqslant R\big(\big\|u(\mu, \lambda') - u_1\big\|^{\beta} + \big\|u(\mu, \lambda) - u_2\big\|^{\beta}\big).$$

Therefore, by (4), we deduce that

$$m\big\|u(\mu, \lambda) - u(\mu, \lambda')\big\|^{\alpha} \leqslant 2RL^{\beta}\big(\|\lambda' - \lambda\|\big)^{\beta\xi}.$$

Step III. We are now ready to complete the proof of our main result. So, let $\mu, \mu' \in U_1$ and $\lambda, \lambda' \in V_1$. We easily see from Steps I and II that

$$\begin{aligned}\big\|u(\mu, \lambda) - u(\mu', \lambda')\big\| &\leqslant \big\|u(\mu, \lambda) - u(\mu', \lambda)\big\| + \big\|u(\mu', \lambda) - u(\mu', \lambda')\big\| \\ &\leqslant k_1\|\mu - \mu'\|^{1/(\alpha-\delta)} + k_2\|\lambda - \lambda'\|^{\beta\xi_1/\alpha}.\end{aligned}$$

Where $k_2 = (2RL^{\beta}/m)^{1/\alpha}$ and $k_1 = (\theta/m)^{1/(\alpha-\delta)}$. The solution $u(\mu, \lambda)$ to $(EP_{\mu,\lambda})$ is then unique for each $\mu \in U_1$ and each $\lambda \in V_1$. The proof is therefore complete. □

Remark 2.2.2.

(1) Notice that following [23, Corollary 32.35], when the equilibria are particularized as solutions to variational inequalities, Theorem 2.2.1 holds true if we substitute (H_0) with the following Aubin property: for neighborhoods $\bar{W}$ of $\bar{u}$ and $\bar{M}$ of $\bar{\lambda}$ and some constant $L > 0$ such that

$$\mathcal{K}_{\lambda} \cap \bar{W} \subset \mathcal{K}_{\lambda'} + L\|\lambda - \lambda'\| B_X \quad \text{for all } \lambda, \lambda' \in \bar{M}.$$

It is worth noting that the classical polyhedral sets have the Aubin property (see [19]). It merits to mention as well in this case that assumptions (H_0)–(H_3) are only needed to be verified in a neighborhood of $\bar{u}$.

(2) A special particular case on the parameters is when we take $\alpha = 2$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, and $\xi = 1$. In this instance the assumptions (H_1) and (H_2) amounts to usual boundedness and Lipschitz behaviour of the bifunction φ. This choice of coefficients will be used in the analysis of our assumptions (see Remark 2.2.3 below).

Remark 2.2.3. Let us now list and analyze some special cases where the conditions of the main result hold:

(i) If we set $f(u, v, \mu) = a(u, v - u, \mu)$ where a is the bilinear form already defined, it is a simple matter to see that condition (H_1) is nothing else but the usual coerciveness and while assumption (H_2) is easily satisfied, the assumption (H_3) upon f amounts to the following condition:

$$\left|a(u,v,\mu)-a(u,v,\mu')\right| \leqslant \theta_1\|\mu-\mu'\|\|v\| \tag{8}$$

for all $u,v\in H$ and all $\mu,\mu'\in\bar{U}$. We emphasize that condition (8) is relaxed form of that considered in [22].

(ii) Let $D\times\Lambda : H\to H^*$ be a c-Lipschitz, that is

$$\left\|D(u,\mu)-D(u',\mu')\right\| \leqslant c\big(\|u-u'\|+\|\mu-\mu'\|\big).$$

Then the bifunction h given by

$$h(u,v,\mu)=\big\langle D(u,\mu),v-u\big\rangle$$

satisfies the condition (H_1). It is worth noting that when

$$h(u,v,\mu)=(l,v-u) \quad \text{or} \quad h(u,v)=\varphi(v)-\varphi(u),$$

where $l\in X^*$ and φ is an extended real valued function on X, the condition (H_1) is trivially verified with $c=0$.

(iii) When we consider an operator $T:V\to V^*$ and its associated bifunction

$$f(u,v)=\langle Tu,v-u\rangle,$$

assumption (H_2) amounts to boundedness condition

$$\sup_{u\in X,\mu\in N}\left\|T(u,\mu)\right\| < +\infty. \tag{9}$$

It is worth pointing out that the above condition is fulfilled whenever for each $u\in X$, the map $\mu\to T(u,\mu)$ is locally Lipschitz and

$$\sup_{u\in X}\left\|T(u,\mu)\right\| < +\infty \tag{10}$$

for all $\mu\in N$. We sum up this by saying that $T(\,.\,,.\,)$ is locally bounded.

(iv) If we take back the preceding example, then assumption (H_3) is verified whenever $\mu\to T(u,\mu)$ is locally Lipschitz continuous. In fact, this could be seen if we write

$$\begin{aligned}\left|f(u,v,\mu)-f(u,v,\mu')\right| &= \left|\big\langle T(u,\mu)-T(u,\mu'),v-u\big\rangle\right| \\ &\leqslant \left\|T(u,\mu)-T(u,\mu')\right\|\|v-u\|.\end{aligned}$$

(v) Assumption (H_4) goes in its spirit to condition of Noor [10], who used it in connection with existence theory for the class of variational inequalities that he first introduced and also contributed to elaborate a fundamental result—see [10, Lemma 3.1]—that have been subsequently a central argument in solution sensitivity of variational inequality investigated by Yen and Lee [22].

Remark 2.2.4. Notice that

(i) No matter how varied the formulations of variational inequalities treated in references mentioned earlier ([4,9,14,21,22] for instance) are, they can be regarded as, and actually are, mixed equilibrium problem for appropriate bifunctions. In the context of this paper we only indicate how our result can be applied to obtain similar result for quasi-variational inclusions studied by Noor and Noor [14] and by Moudafi and Noor [14]: let H be an Hilbert space, $T : H \to H$ be single-valued operator, $A : H \times H \to 2^H$ be a multivalued operator, and consider corresponding quasi-variational inclusion

$$\exists u \in H \quad \text{such that} \quad 0 \in Tu + A(u,u). \tag{QVI}$$

If we consider the bifunction defined by

$$\varphi(u,v) = \langle Tu, v-u\rangle + \inf_{\xi \in A(v,u)} \langle \xi, v-u\rangle,$$

we can see that (QVI) can be reduced to problem (EP) under maximal monotonicity of A and other normal assumptions. A forthcoming note will be devoted to quasi-variational inclusions with additional constraints (normal cone) where the operator A is not necessarily maximal.

(ii) Notice that our result is not affected if we consider the parameters μ and λ in metric spaces.

3. Conclusion

We have studied sensitivity analysis for abstract equilibrium problems in normed spaces, we point out here—as it was explained in the papers by Blum and Oettli [1], by Noor [13], and by Noor and Noor [14]—that the mixed equilibrium problems formulation constitutes an unified approach for studying a wide class of variational problems beyond that we have treated here.

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